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A characterization of injective modules is developed for which it is shown, that with certain restrictions, a module is injective if it is injective with respect to the unique minimal essential left ideal of the ring. Using this characterization, the long column left ideal, A_n, of the $n \times n$ upper triangular matrix ring over a field F, $UT_n(F)$, is shown to be $UT_n(F)$ - injective.

From this it follows that the column left ideals of the full $n \times n$ matrix ring over F, $M_n(F)$, which are $UT_n(F)$ - isomorphic to A_n, are injective. Hence $M_n(F)$ is $UT_n(F)$ - injective. As $M_n(F)$ is an essential extension of $UT_n(F)$, then it is isomorphic to the injective envelope of $UT_n(F)$.

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THE INJECTIVE ENVELOPE OF THE $n \times n$ UPPER TRIANGULAR MATRIX RING OVER A FIELD

by

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to prove that the full $n \times n$ matrix ring over a field F is the injective envelope of the upper triangular $n \times n$ matrix ring over F.

In order that this paper be self-contained, Chapter II is devoted to the basic definitions and properties of modules. However, it has been assumed that the reader has a knowledge of the basic properties of groups and rings. Known theorems have been stated without proof but with a reference as to where the reader may find a proof.

Injective modules and essential submodules are discussed in Chapter III, and some distinctive properties of injective modules are developed. This discussion culminates in showing that with certain restrictions we have a characterization of injective modules, due to K. A. Byrd [Theorem 3.12], that to our knowledge is not found in the literature.

Chapters IV and V are devoted to an investigation of the properties of the upper triangular matrix ring and the full matrix ring over a field F. We show, using the results established in Chapter III, that the full matrix ring is injective, which is the major step in our argument. Then using a characterization of injective envelopes discussed in Chapter V, we conclude this paper by establishing that the full matrix ring satisfies the properties of the injective envelope of the upper triangular matrix ring.

CHAPTER II

PRELIMINARIES

The following basic definitions and properties of modules and matrices will be used extensively throughout this paper and are therefore listed here.

Definition 2.1. If R is a ring with unity, a unital left R-module, denoted $_RM$, is an abelian group $(M, +)$ together with a function $\mu : R \times M \to M$, where $\mu((r, m)) = r \mu m$, such that the following properties are satisfied:

i) $r_1 \mu(m_1 + m_2) = r_1 \mu m_1 + r_1 \mu m_2$,

ii) $(r_1 + r_2) \mu m_1 = r_1 \mu m_1 + r_2 \mu m_1$,

iii) $(r_1 \cdot r_2) \mu m_1 = r_1 \mu(r_2 \mu m_1)$,

iv) $1 \mu m_1 = m_1$,

for all $r_1, r_2 \in R$, $m_1, m_2 \in M$.

An example of a unital left R-module is a left ideal of the ring R, where the function μ is simply left multiplication. The reader can easily see how one would define a unital right R-module, M_R, where $\mu : M \times R \to M$. Usually the notation for the function μ is surpressed and $r \mu m$ is written rm. As we will be considering only rings with unity, it is understood that the terms "ring" and "module" will mean "ring with unity" and "unital left R-module", respectively.

Definition 2.2. A submodule ${}_R N$ of a module ${}_R M$ is a subgroup N of M with the property that for each $n \in N$ and $r \in R$, $r\,n \in N$.

The module ${}_R R$ is called the (left) regular R-module. One can easily see that the submodules of ${}_R R$ are identically the left ideals of R. It can be shown that the intersection of submodules of ${}_R M$ is again a submodule of ${}_R M$.

Remark 2.3. Given a family $\{M_i\}_I$ of R-modules, we recall the direct sum, $\oplus\Sigma_I M_i$, is defined by

$\oplus\Sigma_I M_i = \{< f(i) > \mid f(i) = 0$ for all but a finite number of $i \in I\}$,

where if $f \in \oplus\Sigma_I M_i$, then $f = < f(i) >$ and $f(i)$ denotes the i^{th} component of f. We can see that $\oplus\Sigma_I M_i$ is an R-module by defining $(a \cdot f)(i) = a \cdot f(i)$. As an example, let $I = \{1, 2\}$; we have $M_1 \oplus M_2$ is a module with the definition $a(x_1, x_2) = (ax_1, ax_2)$, recalling that $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$.

We can also define the canonical injection map

$\theta_k : M_k \to \oplus\Sigma_I M_i$ by $\theta_k(m_k) = < m_k >$ where $< m_k > (i) = \begin{cases} m_k & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$

The canonical projection map $\pi_k : \oplus\Sigma_I M_i \to M_k$ is naturally defined by $\pi_k(< m_i >) = m_k$.

Definition 2.4. If ${}_R M$ and ${}_R N$ are modules, then a function $f : {}_R M \to {}_R N$ is an R-homomorphism provided

$$f(m_1 + m_2) = f(m_1) + f(m_2)$$

and

$$f(rm_1) = rf(m_1),$$

for all $r \in R$, $m_1, m_2 \in M$. The set of all R-homomorphisms from ${}_RM$ to ${}_RN$ is denoted by $\mathrm{Hom}_R(M, N)$.

If ${}_RN$ is a submodule of ${}_RM$, we will denote the canonical injection map from ${}_RN$ to ${}_RM$ by θ_N, $\theta_N(n) = n$. If ${}_RM = {}_RN \oplus {}_RK$, then π_N denotes the canonical projection map from ${}_RM$ to ${}_RN$. Clearly θ_N and π_N are R-homomorphisms.

Definition 2.5. If $f \in \mathrm{Hom}_R(M, N)$, then f is an R-isomorphism if there exists a $g \in \mathrm{Hom}_R(N, M)$ such that $f \circ g = 1_N$ and $g \circ f = 1_M$.

It is easy to see that f is an R-isomorphism if and only if f is both one to one and onto.

Definition 2.6. A sequence of R-homomorphisms $< f_i >_I$ is said to be an exact sequence if for

$$\cdots \xrightarrow{f_{i-1}} K \xrightarrow{f_i} L \xrightarrow{f_{i+1}} M \xrightarrow{f_{i+2}} N \xrightarrow{f_{i+3}} \cdots$$

$\ker f_{i+1} = \mathrm{im}\, f_i$ for all $i \in I$.

Therefore, the sequences $0 \to K \xrightarrow{f} L$ and $M \xrightarrow{h} N \to 0$ are exact if and only if f is one to one and h is onto.

Definition 2.7. We say the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & S \arrow[d, "\beta"] \\ & T \end{tikzcd}$$

commutes, or is a <u>commutative</u> <u>diagram</u> if $\gamma = \beta\alpha$, where R, S, and T are R-modules and α, β, and γ are R-homomorphisms.

If F is a field and n is a positive integer, we assume that the reader is familiar with the full $n \times n$ matrix ring over F. We denote this ring by $M_n(F)$.

<u>Definition 2.8</u>. A matrix unit, denoted E_{ij}, is a matrix in $M_n(F)$ which has 1 in the i^{th} row, j^{th} column and zero elsewhere. We denote the zero matrix by (0).

For the matrix units E_{ij}, E_{rs} we have the following properties [4, p. 14]:

$$\text{i)}\quad E_{ij}\, E_{rs} = \begin{cases} (0), & \text{if } j \neq r \\ E_{is} & \text{if } j = r, \end{cases}$$

and

$$\text{ii)}\quad a\, E_{ij}\; b\, E_{rs} = \begin{cases} (0) & \text{if } j \neq r \\ ab\, E_{is} & \text{if } j = r \end{cases} \quad \text{where } a, b \in F.$$

We remark that the full $n \times n$ matrix ring over a field F, $M_n(F)$, can be represented by

$$M_n(F) = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, E_{ij} \;\middle|\; a_{ij} \in F \right\}.$$

We will denote the subring of $M_n(F)$ which has elements of F only in the k^{th} column of $M_n(F)$ by A_k; thus

$$A_k = \left\{ \sum_{i=1}^{n} a_{ik}\, E_{ik} \;\middle|\; a_{ik} \in F \right\}.$$

It is not difficult to see that A_k is a left ideal of $M_n(F)$ for each $k = 1, \cdots, n$. We will refer to A_k as a <u>column</u> <u>left</u> <u>ideal</u> of $M_n(F)$.

Definition 2.9. The upper triangular $n \times n$ matrix ring over a field F, $UT_n(F)$, is the following subring of $M_n(F)$:

$$UT_n(F) = \left\{ \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij} \;\middle|\; a_{ij} \in F \right\}.$$

The reader will observe that if $i > j$, the component a_{ij} is always zero.

Note that we have the left ideal A_n common to both of the rings $M_n(F)$ and $UT_n(F)$, for a fixed positive integer n, where

$$A_n = \left\{ \sum_{i=1}^{n} a_{in} E_{in} \;\middle|\; a_{in} \in F \right\}.$$

It can be shown that A_n is a two-sided ideal of $UT_n(F)$ whereas it is only a left ideal of $M_n(F)$. We shall call A_n, when referring to it as an ideal of $UT_n(F)$, the long column ideal of $UT_n(F)$.

Using the ordinary definitions for addition and multiplication of matrices, it is straightforward to show that

$$(2.10) \quad \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij} \sum_{r=1}^{n} b_{rk} E_{rk} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{jk} E_{ik}.$$

CHAPTER III

ESSENTIAL LEFT IDEALS AND INJECTIVE MODULES

Definition 3.1. A submodule ${}_RN$ of ${}_RM$ is said to be an essential submodule of ${}_RM$ if for every submodule ${}_RL \neq 0$ of ${}_RM$, ${}_RN \cap {}_RL \neq 0$.

In case ${}_RN$ is an essential submodule of ${}_RM$, then we say ${}_RM$ is an essential extension of ${}_RN$.

Proposition 3.2. A module ${}_RM$ is an essential extension of ${}_RN$ if and only if $Rx \cap N \neq 0$ for each $0 \neq x \in M$.

Proof: ($\rightarrow$) Assume ${}_RN$ is essential in ${}_RM$. For each $0 \neq x \in M$ it is easy to see Rx is a nonzero submodule of ${}_RM$ so $Rx \cap N \neq 0$.

($\leftarrow$) Suppose for all $0 \neq x \in M$, $Rx \cap N \neq 0$. Let $0 \neq {}_RL$ be any submodule of ${}_RM$. If $0 \neq k \in L$, then Rk is contained in L, as L is an R-module. By hypothesis $Rk \cap N \neq 0$. Thus $L \cap N \neq 0$ and N is essential in M.

Definition 3.3. If ${}_RK$ is a submodule of ${}_RM$, then a submodule ${}_RN$ of ${}_RM$ maximal with respect to ${}_RN \cap {}_RK = 0$ is called an M-complement of ${}_RK$.

It is easy to see, using Zorn's Lemma, that for every submodule ${}_RK$ of ${}_RM$ there exists an M-complement of ${}_RK$.

Theorem 3.4. Let ${}_RN$ and ${}_RK$ be submodules of ${}_RM$. If ${}_RN$ is an M-complement of ${}_RK$, then $N + K$ is essential in M [2, p. 16].

Note that $N + K$ is a direct sum by its construction.

<u>Definition 3.5</u>. A module ${}_RM$ is <u>R-injective</u> if and only if for every exact sequence of R-modules $0 \to L \xrightarrow{f} N$, that is the R-homomorphism $f : {}_RL \to {}_RN$ is one to one, and for every $\phi \in \mathrm{Hom}_R(L,M)$ there exists a $\psi \in \mathrm{Hom}_R(N,M)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\phi"'] & N \arrow[dl, "\psi"] \\ & M & \end{tikzcd}$$

commutes, that is $\phi = \psi f$.

There is no loss in generality in assuming ${}_RL$ is a submodule of ${}_RN$ and letting f be θ_L, the canonical injection map. We will now show that in order to test for injectivity, it is enough to consider essential submodules.

<u>Theorem 3.6</u>. A module ${}_RM$ is R-injective if and only if for each module ${}_RN$ and for each essential submodule ${}_RK$ of ${}_RN$, given $\phi \in \mathrm{Hom}_R(K,M)$ there exists a $\psi \in \mathrm{Hom}_R(N,M)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\theta_K"] \arrow[d, "\phi"'] & N \arrow[dl, "\psi"] \\ & M & \end{tikzcd}$$

commutes, i.e., $\phi = \psi\,\theta_K$.

<u>Proof</u>: ($\to$) If ${}_RM$ is R-injective then the diagram commutes by definition.

($\leftarrow$) Let ${}_RL$ be any submodule of ${}_RN$ and consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\theta_L"] \arrow[d, "\phi'"'] & N \\ & M & \end{tikzcd} \qquad (1)$$

By Zorn's Lemma, there exists a submodule $_RH$ contained in $_RN$ such that H is the R-complement of L. Then by Theorem 3.4 $L + H$ is essential in N.

Define $\phi : L + H \to M$ by $\phi = \phi' \pi_L$. It is clear that $\phi \in \mathrm{Hom}_R(L + H, M)$, as ϕ' and π_L are R-homomorphisms. By hypothesis, there exists a $\psi \in \mathrm{Hom}_R(N,M)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L + H \arrow[r, "\theta_{L+H}"] \arrow[d, "\phi"'] & N \arrow[dl, "\psi"] \\ & M & \end{tikzcd} \qquad (2)$$

commutes, where θ_{L+H} is the canonical injection. We claim that ψ is the R-homomorphism for which diagram (1) commutes. For if $k \in L$, then $\psi\theta_L(k) = \psi(k) = \psi\theta_{L+H}(k) = \phi(k) = \phi'\pi_L(k) = \phi'(k)$. Thus for each $\phi' : L \to M$ there exists a ψ for which the diagram commutes.

<u>Theorem 3.7</u>. (Baer's Lemma)

For a module $_RM$, the following are equivalent:

a) $_RM$ is an injective R-module.

b) For every left ideal I of R and $\phi \in \mathrm{Hom}_R(I,M)$, there exists an R-homomorphism $\psi : R \to M$ such that $\psi(a) = \phi(a)$ for each $a \in I$.

c) For every left ideal I of R and for every R-homomorphism $\phi : I \to M$ there exists an $m \in M$ such that

$\phi(a) = am$ for all $a \in I$.

For a proof of Theorem 3.7 see [5, p. 25].

The reader will see that (b) simply says for the exact sequence $0 \to I \to R$, where I is both a left ideal of R and a left submodule of ${}_RR$, and $\phi \in \mathrm{Hom}_R(I,M)$, there exists a $\psi \in \mathrm{Hom}_R(R,M)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\theta_I"] \arrow[d, "\phi"'] & R \arrow[dl, "\psi"] \\ & M & \end{tikzcd}$$

commutes.

Therefore, using essentially the same argument as that of Theorem 3.6, we have the following corollary.

<u>Corollary 3.8</u>. A module ${}_RM$ is R-injective if and only if for every essential left ideal I of R and $\phi \in \mathrm{Hom}_R(I,M)$ there exists an R-homomorphism $\psi : R \to M$ such that $\psi(a) = \phi(a)$ for every $a \in I$.

We can see that if $\psi \in \mathrm{Hom}_R(R,M)$, then $\psi(r) = r\psi(1)$ for every $r \in R$. Conversely left multiplication of a fixed element $x \in M$ by elements of R, is an element of $\mathrm{Hom}_R(R,M)$. Therefore ${}_RM$ is R-injective, if for every essential left ideal L of R and $\phi \in \mathrm{Hom}_R(L,M)$ there exists an $m \in M$ such that $\phi(k) = km$ for each $k \in L$.

<u>Defintion 3.9</u>. For a subset S of a module ${}_RM$,

$(0 : S) = \{r \in R | rx = 0$ for each $x \in S\}$. Also

$(S : M) = \{r \in R | rx \in S$ for each $x \in M\}$.

We can easily see that for $x \in {}_RM$, $(0:x)$ is a left ideal of R.

Lemma 3.10. If ${}_RS$ and ${}_RN$ are submodules of ${}_RR$, with S an essential left ideal of N, and N essential in R, then for any $n \in N$, $(S : n)$ is an essential left ideal of R.

Proof: We can easily see $(S : n)$ is a left ideal of R. Therefore, let $0 \neq x \in R$ and $n \in N$. As Rxn is a left ideal of N, then $Rxn \cap S \neq 0$ as S is essential in N. Hence $Rx \cap (S : n) \neq 0$ and $(S : n)$ is essential in R.

Definition 3.11. For a module ${}_RM$, let $Z({}_RM) = \{x \in M \mid (0 : x)$ is an essential left ideal of $R\}$. We call $Z({}_RM)$ the singular submodule of ${}_RM$ since it can be shown that $Z({}_RM)$ is a submodule of ${}_RM$. However, $Z({}_RR)$ is a two-sided ideal of R [2, p. 47].

Theorem 3.12. Let ${}_RM$ be any module with zero singular submodule, $Z(M) = 0$, and R be a ring with a unique minimal essential left ideal S. Then ${}_RM$ is injective if and only if for every $\phi' \in \mathrm{Hom}_R(S,M)$ there exists a $\psi \in \mathrm{Hom}_R(R,M)$, such that $\psi\, \theta'_S = \phi'$.

Proof: $(\rightarrow)$ Immediate from Theorem 3.7 (b).

$(\leftarrow)$ Let N be an essential left ideal of R and S be the unique minimal essential left ideal of R. We are given the diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\theta_N"] \arrow[d, "\phi"'] & R \\ & M & \end{tikzcd} \qquad \text{with } \phi \in \mathrm{Hom}_R(N,M).$$

It is not difficult to see that $S \cap N$ is essential in R, so that $S \subseteq N$. As S is essential in N, which is essential in R, we have

$\theta_S : S \to N$ and $\phi' = \phi\theta_S$ is an R-homomorphism from S to M. By hypothesis there exists an $m \in M$ such that $\phi\theta_S(s) = sm$ for all $s \in S$ where $m = \psi(1)$. Using Lemma 3.8, for each $n \in N$, $(S : n)$ is an essential left ideal of R. It is easy to see that $(S : n) \cdot n$ is contained in S, as we recall $(S : n)$ $= \{r \in R | r \cdot n \in S\}$. Therefore,

(1) $\phi((S : n) \cdot n) = \phi\theta_S((S : n) \cdot n) = [(S : n) \cdot n] \cdot m.$

However as ϕ is an R-homomorphism, then

(2) $\phi((S : n) \cdot n) = (S : n)\phi(n).$

From (1) and (2) we have

$$(S : n)\phi(n) = [(S : n) \cdot n] \cdot m,$$

or

$$(S : n)(\phi(n) - nm) = 0.$$

Recall that $Z(M) = 0$; that is if $m \in M$, $km = 0$ for all $k \in K$, and K is an essential left ideal of R, then $m = 0$. As $\phi(n) - nm \in M$ and $(S : n)$ is an essential left ideal of R then $\phi(n) - nm = 0$. Hence $\phi(n) = nm$ for every $n \in N$. This implies, by Theorem 3.7 (c) and Corollary 3.8, that ${}_RM$ is injective.

CHAPTER IV

THE LONG COLUMN IDEAL IS INJECTIVE OVER $UT_n(F)$

We will show that the upper triangular matrix ring contains a unique minimal essential left ideal and that $Z(A_n) = (0)$, where A_n denotes the long column ideal of $UT_n(F)$. We will then be able to use the results developed in Chapter III to conclude A_n is $UT_n(F)$ - injective.

Lemma 4.1. Every essential left ideal of $UT_n(F)$ contains the left ideal

$$S = \left\{ \sum_{j=1}^{n} a_{1j} E_{1j} \;\middle|\; a_{1j} \in F \right\}.$$

Proof: Let $B_j = \{a_{1j} E_{1j} \mid a_{1j} \in F\}$. Then each B_j is a simple left ideal of $UT_n(F)$ and $S = B_1 + \cdot \cdot \cdot + B_n$. Let N be any essential left ideal of $UT_n(F)$. Then $N \cap B_j \neq 0$ and as B_j is simple, $N \cap B_j = B_j$. Therefore $B_j \subseteq N$ for each $j = 1, \cdot \cdot \cdot, n$. Hence $S = B_1 + \cdot \cdot \cdot + B_n \subseteq N$.

Theorem 4.2. The upper triangular matrix ring over a field F has a unique minimal essential left ideal which is contained in every essential left ideal.

Proof: As

$$S = \left\{ \sum_{j=1}^{n} a_{1j} E_{1j} \;\middle|\; a_{1j} \in F \right\}$$

is contained in every essential left ideal of $UT_n(F)$ by Lemma 4.1, it is enough to show that S is also an essential left ideal of $UT_n(F)$.

Let L be a nonzero left ideal of $UT_n(F)$. Then there exists at least one column of L for which there exists a nonzero component. Choose any nonzero column, say k, of L. It is easy to see that the component a_{1k} is not always zero; for given any matrix X with $0 \neq a_{ik}$, multiplying X on the left by E_{1i} we get an element in L which has a nonzero a_{1k} component. Therefore $S \cap L \neq 0$ and we may conclude S is essential in $UT_n(F)$. As every essential left ideal of $UT_n(F)$ contains S, then S is minimal.

Theorem 4.3. The upper triangular matrix ring over F has zero singular ideal, or $Z(UT_n(F)) = (0)$.

Proof: Suppose $Z(UT_n(F)) \neq (0)$. Then there exists $(0) \neq A \in UT_n(F)$ such that $(0 : A)$ is an essential left ideal of $UT_n(F)$. Recall that $(0 : A) = \{A' \in UT_n(F) \mid A'A = (0)\}$. As $(0 : A)$ is essential in $UT_n(F)$, then S, the unique minimal essential left ideal of $UT_n(F)$, is contained in $(0 : A)$ by Theorem 4.2. Thus for any element U' of S, $U'A = (0)$.

Consider the set U, where $U = \{U_k' \mid U_k' = E_{1k},\ 1 \leq k \leq n\}$, which is clearly a subset of S. Then for each $U_k' \in U$, it follows that $U_k'A = (0)$. Let A be represented by

$$A = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij}.$$

Then for any fixed k we have

$$U_k'A = E_{1k} \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{1k} E_{ij}$$

$$= \sum_{j=k}^{n} a_{kj} E_{1j}$$

and

$\sum_{j=k}^{n} a_{kj} E_{1j} = (0)$. Hence for all $j = k, \cdots, n$ $a_{kj} = 0$. Thus $a_{kj} = 0$ for $1 \leq k \leq n$. This says $a_{ij} = 0$ for all i, j and

$$A = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E_{ij} = (0),$$

a contradiction to the choice of A. Thus $Z(UT_n(F)) = (0)$.

<u>Lemma 4.4</u>. If $_RK$ is a submodule of $_RM$, then $Z(K) = K \cap Z(M)$.

<u>Proof</u>: Let $x \in Z(K)$ then by definition $x \in K$, hence $x \in M$, and $(0 : x)$ is essential in R. Thus $x \in Z(M)$. Conversely, if $x \in K \cap Z(M)$, then $x \in K$ and $(0 : x)$ is essential in R, as $x \in Z(M)$. Thus $x \in Z(K)$.

In particular, as $Z(UT_n(F)) = (0)$ and A_n is contained in $UT_n(F)$, then $Z(A_n) = (0)$.

<u>Theorem 4.5</u>. The long column left ideal A_n of $UT_n(F)$ is injective as a $UT_n(F)$ - module.

<u>Proof</u>: It has been established that

$$S = \left\{ \sum_{j=1}^{n} a_{1j} E_{1j} \;\middle|\; a_{1j} \in F \right\}$$

is the unique minimal essential left ideal of $UT_n(F)$. By Theorem

3.11 it suffices to show that any $\phi \in \mathrm{Hom}_{UT_n(F)}(S, A_n)$ can be extended to a $UT_n(F)$ - homomorphism from $UT_n(F)$ to A_n. We recall

$$A_n = \left\{ \sum_{i=1}^{n} x_{in} E_{in} \;\middle|\; x_{in} \in F \right\}.$$

Let $\phi : S \to A_n$ be a $UT_n(F)$ - homomorphism; we define

$$\phi(E_{1j}) = \sum_{i=1}^{n} {}_j x_{in} E_{in} .$$

Let $A = \sum_{j=1}^{n} a_{1j} E_{1j}$ be any element of S. Then

$$\phi(A) = \phi\left(\sum_{j=1}^{n} a_{1j} E_{1j}\right) = \sum_{j=1}^{n} \phi(a_{1j} E_{1j})$$

$$= \sum_{j=1}^{n} \phi(a_{1j} E_{11} E_{1j}) \quad \text{by 2.5(i),}$$

and

$$\phi(A) = \sum_{j=1}^{n} a_{1j} E_{11} \phi(E_{1j})$$

$$= \sum_{j=1}^{n} a_{1j} E_{11} \sum_{i=1}^{n} {}_j x_{in} E_{in}$$

$$= \sum_{j=1}^{n} a_{1j} {}_j x_{1n} E_{1n} \quad \text{by 2.8 (ii).}$$

Using 2.8 (i), we can rewrite this as

$$\phi(A) = \sum_{j=1}^{n} a_{1j} {}_j x_{1n} E_{1j} E_{jn},$$

and then using 2.8 (ii), we have

$$\phi(A) = \sum_{j=1}^{n} a_{1j} E_{1j} \sum_{k=1}^{n} {}_k x_{1n} E_{kn}$$

$$= A \cdot \sum_{k=1}^{n} {}_k x_{1n} \, E_{kn} \quad \text{where}$$

$$Y = \sum_{k=1}^{n} {}_k x_{1n} \, E_{kn} \in A_n.$$

Therefore for any $A \in S$, we have $Y \in A_n$ such that $\phi(A) = A \cdot Y$. Now define $\psi : UT_n(F) \to A_n$ by $\psi(B) = B \cdot Y$ for $B \in UT_n(F)$. It is easy to see that ψ is a $UT_n(F)$ - homomorphism which extends ϕ.

CHAPTER V

THE INJECTIVE ENVELOPE OF $UT_n(F)$

This section will be devoted to showing that the full $n \times n$ matrix ring over a field F, $M_n(F)$, is injective as a $UT_n(F)$ - module and is also an essential extension of the upper triangular $n \times n$ matrix ring over a field F. This is equivalent to showing $M_n(F)$ is the injective envelope of $UT_n(F)$.

Theorem 5.1. If A_k is a column left ideal of $M_n(F)$, then A_k is $UT_n(F)$ - isomorphic to A_n.

Proof: Let A_k represent any column left ideal of $M_n(F)$. Define $\alpha : A_k \to A_n$ by

$\alpha\left(\sum_{i=1}^{n} a_{ik} E_{ik}\right) = \sum_{i=1}^{n} a_{ik} E_{in}$. It is clear that α is well defined. We need to show α is a $UT_n(F)$ - homomorphism.

Let $\sum_{i=1}^{n} a_{ik} E_{ik}$, $\sum_{i=1}^{n} b_{ik} E_{ik} \in A_k$.

Then

$$\begin{aligned}
\alpha\left(\sum_{i=1}^{n} a_{ik} E_{ik} + \sum_{i=1}^{n} b_{ik} E_{ik}\right) &= \alpha\left(\sum_{i=1}^{n} (a_{ik} + b_{ik})E_{ik}\right) \\
&= \sum_{i=1}^{n} (a_{ik} + b_{ik})E_{in} \\
&= \sum_{i=1}^{n} a_{ik} E_{in} + \sum_{i=1}^{n} b_{ik} E_{in}
\end{aligned}$$

5.1 (i) $$= \alpha\left(\sum_{i=1}^{n} a_{ik} E_{ik}\right) + \alpha\left(\sum_{i=1}^{n} b_{ik} E_{ik}\right).$$

Also if $\sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} E_{ij}$ is any element of $UT_n(F)$, then using 2.10, 2.8 (i) and 5.1 (i), we have

$$\alpha\left(\sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} E_{ij} \sum_{r=1}^{n} a_{rk} E_{rk}\right) = \alpha\left(\sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} a_{jk} E_{ik}\right)$$

$$= \alpha\left(\sum_{j=1}^{n} u_{1j} a_{jk} E_{1k} + \sum_{j=2}^{n} u_{2j} a_{jk} E_{2k} + \cdots + \sum_{j=n}^{n} u_{nj} a_{jk} E_{nk}\right)$$

$$= \alpha\left(\sum_{j=1}^{n} u_{1j} a_{jk} E_{1k}\right) + \alpha\left(\sum_{j=2}^{n} u_{2j} a_{jk} E_{2k}\right) + \cdots + \alpha\left(\sum_{j=n}^{n} u_{nj} a_{jk} E_{nk}\right)$$

$$= \sum_{j=1}^{n} u_{1j} a_{jk} E_{1n} + \sum_{j=2}^{n} u_{2j} a_{jk} E_{2n} + \cdots + \sum_{j=n}^{n} u_{nj} a_{jk} E_{nn}$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} a_{jk} E_{in}$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} E_{ij} \sum_{r=1}^{n} a_{rk} E_{rn}$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} u_{ij} E_{ij} \, \alpha\left(\sum_{r=1}^{n} a_{rk} E_{rk}\right).$$

This allows us to conclude α is a $UT_n(F)$ - homomorphism.

It is easy to see α is onto. To show α is one to one we consider

$$\ker \alpha = \left\{ \sum_{r=1}^{n} a_{rk} E_{rk} \;\middle|\; \alpha\left(\sum_{r=1}^{n} a_{rk} E_{rk}\right) = (0) \right\}.$$

Equivalently,

$$\ker \alpha = \left\{ \sum_{r=1}^{n} a_{rk} E_{rk} \;\middle|\; \sum_{r=1}^{n} a_{rk} E_{rn} = (0) \right\}.$$

This implies $a_{rk} = 0$ for all $r = 1, \cdots, n$ or $\ker \alpha = (0)$.

Hence α is a $UT_n(F)$ - isomorphism.

Therefore, A_k is $UT_n(F)$ - isomorphic to A_n for $k = 1, \cdots, n$. As A_n is $UT_n(F)$ - injective then each A_k is $UT_n(F)$ - injective.

Theorem 5.2. If ${}_RM = {}_RL \oplus {}_RN$, then ${}_RM$ is injective if and only if both ${}_RL$ and ${}_RN$ are injective [1, p. 386].

Obviously $M_n(F) = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Therefore as each A_k is $UT_n(F)$ - injective then $M_n(F)$ is $UT_n(F)$ - injective.

Theorem 5.3. $M_n(F)$, as a $UT_n(F)$ - module, is an essential extension of $UT_n(F)$.

Proof: Let $(0) \neq A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij}$ be any element in $M_n(F)$. As $A \neq (0)$, then there exists a component $a_{rs} \in A$ such that $a_{rs} \neq 0$. Clearly $E_{1r} \in UT_n(F)$. Thus using 2.8 (ii),

$$E_{1r} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij} = \sum_{j=1}^{n} a_{rj} E_{1j}.$$ As $\sum_{j=1}^{n} a_{rj} E_{1j}$ is an element of $UT_n(F)$, then

$$E_{1r} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E_{ij} \cap UT_n(F) \neq (0).$$ Hence by Proposition 3.2, $M_n(F)$ is an essential extension of $UT_n(F)$.

Definition 5.4. The injective hull or injective envelope of a module ${}_RM$, denoted by $E({}_RM)$, is an injective R-module such that if N' is an injective R-module with $M \subseteq N' \subseteq E(M)$, then $N' = E(M)$.

It can be shown that the injective envelope of a module ${}_RM$ can be characterized up to isomorphism as a module which is both an injective R-module and an essential extension of ${}_RM$. Further, $E({}_RM)$ is the smallest injective module containing ${}_RM$, as can be seen from the definition, and the largest essential extension of ${}_RM$ [5, p. 23].

As it has been shown that the full $n \times n$ matrix ring is both injective as a $UT_n(F)$ - module and an essential extension of $UT_n(F)$, then $M_n(F)$ as a $UT_n(F)$ - module is isomorphic to the injective hull of $UT_n(F)$ as a $UT_n(F)$ - module.

SUMMARY

In conclusion, we have shown that the long column ideal, A_n, of $UT_n(F)$ is injective as a $UT_n(F)$ - module, using a characterization of injective modules developed in Chapter III. As there exists a $UT_n(F)$ - isomorphism between A_n and each column left ideal A_k of $M_n(F)$, then each A_k is $UT_n(F)$ - injective. $M_n(F)$ is a direct sum of its column left ideals and is therefore also $UT_n(F)$ - injective. Then knowing $M_n(F)$ is an essential extension of $UT_n(F)$ is enough to complete our argument that $M_n(F)$ is the injective envelope of $UT_n(F)$ of a module over itself.

The reader may well ask at this point if this result can be extended to infinite matrix rings. Though the author is unable to answer this question specifically, it is clear that if the answer is affirmative, the same approval used in this paper cannot be used. Recall that in proving $M_n(F)$ is $UT_n(F)$ - injective we used Theorem 5.2, which states a finite direct sum of R-injective modules is R-injective. This is not necessarily true for an infinite direct sum [5, p. 13].

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